Math 261B Tues. 9/29

$T \subset B \subset G = GL_n \qquad \mathfrak{gl}_n = M_n = \mathfrak{t} \oplus \bigoplus \mathfrak{g}_\alpha$

$$\begin{pmatrix} \ddots & & \cdot \\ & \mathfrak{t} & \\ \cdot & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \qquad , \quad \to \quad \begin{array}{c} \qquad i,j \\ \begin{pmatrix} 1 & & b \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array} = U_\alpha$$

$b \; \underline{t}$

$\alpha = e_i - e_j \longleftrightarrow E_{ij} \in \mathfrak{g}_\alpha$

$t_i / t_j \qquad i \to \begin{pmatrix} & 1 \\ & \end{pmatrix} \; \uparrow j$

$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus k h_\alpha \cong \mathfrak{sl}_2$

T normalizes U_α

$T \ltimes U_\alpha \subset G \qquad b \, \underline{t} \; b' \, \underline{t}'$

$(b, \underline{t}) \cdot (b', \underline{t}') \qquad b \cdot \underline{t} \, b' \, \underline{t}^{-1} \; \underline{t} \, t'$

$\qquad \shortparallel$

$(b + t^\alpha b', \underline{t} \; \underline{t}') \qquad t^\alpha b'$

t_i / t_j

$SL_2 \to G \qquad \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cong \mathbb{G}_m$

$\alpha^\vee \quad U \to U_\alpha$

$SL_2 \supset U \qquad \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \cong \mathbb{G}_a$ unipotent

$\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$

$G \curvearrowright V \qquad T \curvearrowright V \qquad V = \bigoplus_{\lambda \in X} V_\lambda$

$v \in V_\lambda \qquad \underline{t} \cdot v = t^\lambda v$

$U_\alpha v \subset V$ is T-invariant $\qquad \underline{t} \; b v = \underline{t} \, b \, \underline{t}^{-1} \; \underline{t} v$

$\qquad (t^\alpha b) \quad t^\lambda v$

$\underline{t} = (t_1, \dots, t_n)$ $\qquad$ $X = X(T)$ character lattice of T

$X = \mathbb{Z}^n$ $\qquad$ $\lambda \in X$ $\qquad$ $t^\lambda = \lambda(\underline{t})$

$T \to \mathbb{G}_m$

$(\lambda_1, \dots, \lambda_n) \to \underline{t} \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n}$

$U_\alpha \times U_\alpha v \to U_\alpha v$ $\leftarrow$ action is tautologically T-invariant

$(b, w) \to bw$

$(\underline{t}\, b\, \underline{t}^{-1}, \underline{t} w) \mapsto \underline{t}\, bw$

$U_\alpha v \xrightarrow[\rho]{} U_\alpha v \otimes k[z]$ "$\mathcal{O}(U_\alpha)$" $\qquad z(b) = b$

must be T-invariant

$(\underline{t} \cdot z)(b) = z(\underline{t}^{-1} b \underline{t})$

$= z(t^{-\alpha} b) = t^{-\alpha} b$

$= t^{-\alpha} z(b)$

$\underline{t} \cdot z = t^{-\alpha} z$

$v \xrightarrow[\rho]{} \sum_{i \geq 0} v_i \otimes z^i$ $\qquad$ for some $v_i \in U_\alpha v$, finitely many $v_i \neq 0$

$b \cdot v = \sum_i' b^i v_i$

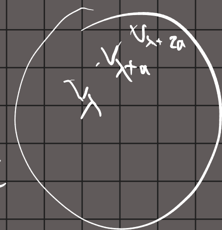
$b \in U_\alpha$

$v \xrightarrow[\rho]{} \sum v_i \otimes z^i$

v: weight λ $\qquad$ v_i: weight $t^{\lambda + \alpha i}$ $\qquad$ z^i: weight $t^{-\alpha i}$

$\underline{t} v = t^\lambda v$ $\qquad$ $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$

$\Rightarrow$ weights of $U_\alpha v$ are in $\lambda + \mathbb{N}\alpha$ (and 1-dimensional).



$\Rightarrow \quad SL_2 \circlearrowright V_\lambda \quad \subseteq \quad \sum_{k\in\mathbb{Z}} V_{\lambda+k\alpha} \qquad V_{\lambda+\mathbb{Z}\alpha}$

$t \xmapsto{\alpha^\vee} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \qquad \langle \alpha^\vee, \lambda + k\alpha \rangle = \langle \alpha^\vee, \lambda \rangle + 2k$

$s \in W(SL_2) = N(T)/T = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \amalg \begin{pmatrix} & -t \\ t^{-1} & \end{pmatrix} \Big/ T \cong S_2 \qquad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

acts on weights $\quad \alpha \mapsto -\alpha$

$s(\lambda) - \lambda = m\alpha \qquad m \in \mathbb{Z}$

$$s(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$$

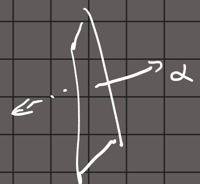
$\langle \alpha^\vee, s(\lambda) - \lambda \rangle = 2m \qquad s(\alpha) = -\alpha$

$= -2\langle \alpha^\vee, \lambda \rangle \qquad \langle \alpha^\vee, s(\lambda) \rangle = -\langle \alpha^\vee, \lambda \rangle$

Fixes $\langle \alpha^\vee, \lambda \rangle = 0 \qquad \alpha \mapsto -\alpha \qquad m = -\langle \alpha^\vee, \lambda \rangle$

$\alpha^\vee = 0$

$X_\mathbb{R} = (\ker \alpha^\vee) \oplus \mathbb{R}\alpha \qquad \mathbb{Z}^n \to \mathbb{Z}$



$X_\mathbb{R} \to \mathbb{R}$

s_α acts as a lattice reflection on X

$s_\alpha^2 = id$

also acts on X^* $\qquad \beta \in X^* \qquad s(\beta) = \beta - \langle \beta, \alpha \rangle \alpha^\vee$

$s_{\alpha,\alpha^\vee}$ on $X \qquad s_{\alpha^\vee,\alpha}$ on X^*

W $N(T)/T$ in G acts on X, contains s_α for each root. (They generate it)

acts on R

GL_n $\quad \alpha = e_i - e_j$

$$\begin{array}{c} \\ i \\ j \end{array} \begin{pmatrix} 1 & & & & \\ & a & & b & \\ & & 1 & & \\ & c & & d & \\ & & & & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & & & & \\ & \cdot & & -t & \\ & & \ddots & & \\ & t^{-1} & & \cdot & \\ & & & & 1 \end{pmatrix}$$

$e_i \leftrightarrow e_j$

$W = S_n \curvearrowright X = \mathbb{Z}^n$

$\curvearrowright R = \{ e_i - e_j \}$

$R_+ \quad e_i - e_j \quad i < j$

$\alpha_i \quad e_i - e_{i+1}$

$\rightsquigarrow$

$e_2 - e_5$
$= e_2 - e_3$
$+ e_3 - e_4$
$+ e_4 - e_5$

$s_i^2 = 1 \qquad s_i s_j = s_j s_i \quad \text{if } |i-j| > 1$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{if } |i-j| = 1$

$s_i = (i, i+1)$ generate $W = S_n$ as a Coxeter

1 2 3
2 1 3
2 3 1
3 2 1

$s_1 s_2 s_1 = s_2 s_1 s_2$

$(\underbrace{s_i s_j s_i \cdots}_{m}) = (s_j s_i \cdots)$

$(s_i s_j)^m = 1$

Cartan Datum Lattice X, X^*, finite $R \subset X$, $R^\vee \subset X^*$

$X \cong \mathbb{Z}^n$ "roots" "coroots"

$\alpha \in R \longleftrightarrow \alpha^\vee \in R^\vee$

s.t. 1) $\langle \alpha^\vee, \alpha \rangle = 2$, so have reflection s_α

2) R is invariant under group W gen. by the s_α

3) only multiples of α in R are $\pm\alpha$

Thm. Cartan Data classify:

- Reductive alg. gps over any alg. closed K
- Reductive Lie gps $\overset{G}{\text{over}}$ $\mathbb{C}$ $=$ semisimple $\times$ $(\mathbb{C}^\times)^n$
- Compact real Lie groups : as $\underset{\text{compact}}{\wedge}$ $\mathbb{R}$ forms of the preceding G

$C \subset G$ Lie$/\mathbb{C}$

Lie$/\mathbb{R}$

$\mathbb{R}$ form $\quad$ $\mathrm{Lie}(G) = \mathrm{Lie}(C) \oplus i\,\mathrm{Lie}(C)$ $\quad$ $\mathrm{Lie}(C)_{/\mathbb{R}} \subset \mathrm{Lie}(G)_{/\mathbb{C}}$

$SO_n(\mathbb{C})$ $\quad AA^T = I \quad M + M^T = 0$

$SO_n(\mathbb{R})$ $\quad AA^T = I \quad M + M^T = 0$

" rotations of unit ball in n space $\quad$ and $\det = 1$

$\det(A)^2 = 1$

$\det(A) = \pm 1$

$U_n \subset GL_n(\mathbb{C})$ is a compact real form.

More classical examples SO_n Sp_n

$SO_n(k) \subset SL_n(k) \curvearrowright k^n$ subgroup preserving a non-degenerate symmetric bilinear form $(,)$ on k^n.

(Disregard char = 2)

Choose $(,)$ so $(e_i, e_j) = \delta_{i+j, n+1}$: $e_1, \dots, e_n$ dual basis to $e_n, \dots, e_1$

Matrix of form is $J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $(x, y) = y^T J x$

A preserves $(,)$ if
$\cap$
SL_n

$(Ay, Ax) = y^T A^T J A x = (y, x) = y^T J x$ $J^2 = I$

$= A^T J A = J$

$J A^T J A = I$

$A A^R = I$ (rather than usual $AA^T = I$)

$A^R = JA^TJ = \begin{pmatrix} & & \\ & A \to A^R & \\ & & \end{pmatrix}$

$$SO_{2n+1} \qquad T = \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix}$$

$$SO_{2n} \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}$$

B = upper Δar subgroups

$$\mathfrak{so}_n = \{ M_n \mid \underbrace{M + M^R = 0}_{\text{why?}} \}$$

$$\begin{pmatrix} x & & \\ & \ddots & \\ & & -x \end{pmatrix}$$